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THERMOELASTIC FRACTURE SOLUTIONS USING DISTRIBUTIONS OF SINGULAR INFLUENCE FUNCTIONS—I

DETERMINING CRACK STRESS FIELDS FROM DISLOCATION DISTRIBUTIONS

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Abstract—A numerical technique has been developed for the determination of stress fields assoicated with a variety of two-dimensional fracturing problems. The procedure allows the analysis of cracks intersecting the free surface of a half-space, of branched or blunted cracks, and of infinitely long arrays of periodically-spaced cracks. The technique employs an efficient surface integral method, using the well-known mathematical equivalence of cracks and distributions of edge dislocations. New conditions are developed for closing the set of equations governing cracks intersecting a free surface, based on a consideration of the stresses at the point of intersection.

INTRODUCTION AND FORMULATION

A method for analyzing quasi-static crack-induced stress fields is developed for two-dimensional crack configurations. The material is assumed to be homogeneous, isotropic, linearly elastic, and either infinite or semi-infinite in extent. For the semi-infinite case, the material occupies the half-space $y \geq 0$ in a rectangular coordinate system. Any (including an infinite) number of cracks may be considered and they may be arbitrarily located. The cracks are piecewise straight and allow the possibility of branch cracks at their tips.

The nomenclature to be used is illustrated by the collection of N cracks shown in Fig. 1. A particular point on crack n is located absolutely by $(x, y)_n = (x_0, y_0)_n + w_n(-\sin\Psi_n, \cos\Psi_n)$ but is uniquely identified simply by w_n. Two additional coordinates, ζ_n and t_n, are defined for each crack n as $(w_n - a_n)/a_n$; these coordinates thus range from -1 to $+1$.

The cracks are represented by continuous distributions of dislocation singularities, as described by Rice[1]. A single edge dislocation is represented by a Burgers vector b_p, equal to the closed integral of $\partial u_p/\partial l$ in the counterclockwise direction around the dislocation (u_p being displacement in the p direction and l being distance along the integration path). The "density" $\mu_p(t)$ of the normalized dislocation distribution along a crack is defined such that $a \cdot \mu_p(t_n)\,\mathrm{d}t_n$ represents the infinitesimal Burgers vector in the p-direction at point t_n. Stresses σ_{jk} at point (x, y) due to the normalized unit dislocation in the p-direction ($b_p/a = 1$) located at point (x_d, y_d) are given by well-known influence functions $\Gamma^p_{jk}([x, y], [x_d, y_d])$[2, 3].

The stress σ_{jk} at point (x, y) is now given by

$$\sigma_{jk}(x, y) = \sum_{n=1}^{N}\sum_{p=1}^{2}\int_{-1}^{1} \mathrm{d}t_n \mu_p(t_n)\Gamma^p_{jk}([x, y], t_n) \tag{1}$$

where p takes on values 1, 2 to signifiy Burgers vectors in the x- and y-directions, respectively. The dislocation densities are determined by requiring that the normal and shear tractions on the crack surfaces equal those specified for the problem. If $\sigma_\nu(\zeta_m)$ and $\sigma_\tau(\zeta_m)$ are the specified normal and shear tractions, respectively, along crack m, then the $\mu_p(t_n)$ must satisfy

$$\begin{Bmatrix}\sigma_\nu(\zeta_m)\\ \sigma_\tau(\zeta_m)\end{Bmatrix} = \sum_{n=1}^{N}\sum_{p=1}^{2}\int_{-1}^{1} \mathrm{d}t_n \mu_p(t_n)\begin{Bmatrix}\Gamma^p_\nu(\zeta_m, t_n)\\ \Gamma^p_\tau(\zeta_m, t_n)\end{Bmatrix} \tag{2}$$

for $m = 1, 2, \ldots, N$. Here the normal and shear influence functions are given (using $i = \sqrt{-1}$)

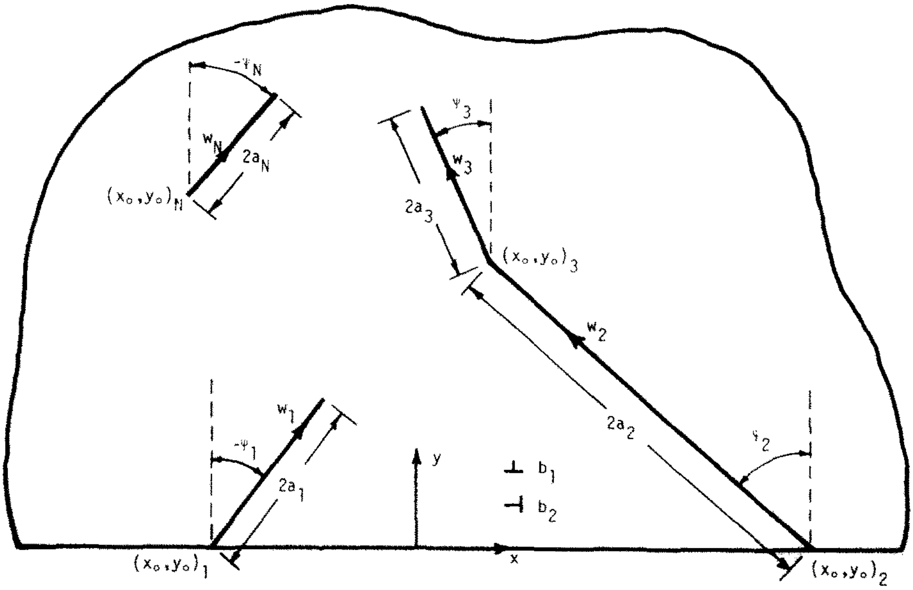

Fig. 1. Nomenclature for crack problems.

by

$$\begin{Bmatrix} \Gamma_\nu^p(\zeta_m, t_n) \\ \Gamma_\tau^p(\zeta_m, t_n) \end{Bmatrix} = \begin{Bmatrix} \mathrm{Re} \\ \mathrm{Im} \end{Bmatrix} [\Gamma_H^p(\zeta_m, t_n) + e^{-2i\Psi_m}\Gamma_D^p(\zeta_m, t_n)] \tag{3a}$$

where

$$\Gamma_H^p(\zeta, t) = \frac{1}{2}[\Gamma_{xx}^p(\zeta, t) + \Gamma_{yy}^p(\zeta, t)] \tag{3b}$$

$$\Gamma_D^p(\zeta, t) = \frac{1}{2}[\Gamma_{xx}^p(\zeta, t) - \Gamma_{yy}^p(\zeta, t)] + i\Gamma_{xy}^p(\zeta, t). \tag{3c}$$

It is known (e.g. [4]) that the dislocation densities for internal cracks have the form

$$\mu_p(t) = f_p(t)/(1-t)^{\alpha_1}(1+t)^{\alpha_2} \tag{4}$$

where $f_p(t)$ is nonsingular and the powers α_1 and α_2 depend on the medium involved; the assumption of homogeneity near all tips ($\alpha_1 = \alpha_2 = 0.5$) is used here.

If the relevant influence functions [2, 3] are written in terms of ζ_m and t_n, they are all found to be of the form

$$\Gamma_{jk}^p(\zeta_m, t_n) = \phi g_{jk}^{mnp}/(\zeta_m - t_n) + \phi h_{jk}^p(\zeta_m, t_n), \tag{5}$$

where $\phi = E/4\pi(1-\nu^2)$, E = Young's modulus, and ν = Poisson's ratio, for an isotropic medium; corresponding forms can be found for an anisotropic medium (see references in [3]). Here g_{jk}^{mnp} is zero for $m \neq n$ and is nonsingular when $m = n$; $h_{jk}^p(\zeta_m, t_n)$ is nonsingular unless $m = n$ and ζ, t both lie at a free surface.

Combining eqns (2), (4) and (5) gives, for $m = 1, 2, \ldots, N$,

$$\begin{Bmatrix} \sigma_\nu(\zeta_m) \\ \sigma_\tau(\zeta_m) \end{Bmatrix} = \phi \sum_{n=1}^{N} \sum_{p=1}^{2} \int_{-1}^{1} \frac{\mathrm{d}t_n f_p(t_n)}{\sqrt{(1-t_n^2)}} \left[\frac{1}{(\zeta_m - t_n)} \begin{Bmatrix} g_\nu^{mnp} \\ g_\tau^{mnp} \end{Bmatrix} + \begin{Bmatrix} h_\nu^p(\zeta_m, t_n) \\ h_\tau^p(\zeta_m, t_n) \end{Bmatrix} \right] \tag{6}$$

where the notation of eqn (3) has been applied to g and h.

Following a procedure such as that outlined in Appendix 2 of Ref. [5], eqn (6) can be reduced to the approximate form

$$\begin{Bmatrix}\sigma_\nu(\zeta_{mr})\\ \sigma_\tau(\zeta_{mr})\end{Bmatrix} \simeq \sum_{n=1}^{N} \frac{\pi}{M_n} \sum_{p=1}^{2} \sum_{k=1}^{M_n} f_p(t_{nk}) \begin{Bmatrix}\Gamma_\nu^p(\zeta_{mr}, t_{nk})\\ \Gamma_\tau^p(\zeta_{mr}, t_{nk})\end{Bmatrix} \tag{7}$$

for $m = 1, 2, \ldots, N$ and $r = 1, 2, \ldots, M_m - 1$.

Equation (7) is a standard matrix equation in which the dislocation density strengths $f_p(t_n)$ are evaluated at the M_n zeroes of T_{M_n}, the Chebyshev polynomials of the first kind, and the crack tractions are specified at the $M_m - 1$ zeroes of U_{M_m-1}, the Chebyshev polynomials of the second kind. Since eqn (7) gives $2\sum_{m=1}^{N}(M_m - 1)$ equations for $2\sum_{n=1}^{N} M_n$ unknowns, $2N$ more equations are needed: two more for each crack. The appropriate equations to be used depend on the type of crack.

Embedded crack

Since a completely embedded crack of length $2a$ must have a determinate amplitude of entrapped dislocations, $a\delta_p$, an additional condition

$$\delta_p = \int_{-1}^{1} \mu_p(t_n)\,\mathrm{d}t_n = \int_{-1}^{1} \frac{f_p(t_n)\,\mathrm{d}t_n}{\sqrt{(1-t_n^2)}} \simeq \frac{\pi}{M_n} \sum_{k=1}^{M_n} f_p(t_k) \tag{8}$$

provides (for $p = 1, 2$) the two equations needed for each crack.

Surface crack

For a crack which intersects the free surface, eqn (8) cannot reasonably be imposed and some other conditions must be sought; these have apparently not been extracted before in the literature. It will be shown that the requirements that σ_{yy} and σ_{xy} vanish and that σ_{xx} remains finite at the point where the crack intersects the free surface leads to a single condition relating the values of the x- and y-components of the dislocation density at the free surface. This condition is satisfied, and satisfactory results are obtained numerically, if the dislocation density is set equal to zero at the free surface.

To obtain this result, a generic surface crack is considered, as shown in Fig. 2. The co-ordinates γ and ω indicate dimensionless distance along the crack from the free surface; $0 \le (\gamma, \omega) \le 1$. Since dislocations which are distant from the point $\gamma = 0$ contribute nothing to σ_{yy} and σ_{xy} at that point and produce bounded σ_{xx} at that point, it is necessary only to consider the single crack in deriving the desired condition on the free surface dislocation density.

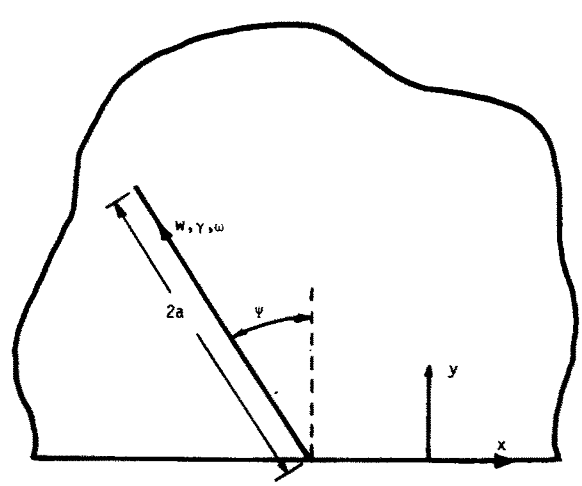

Fig. 2. Generic crack considered in determining equations needed to close numerical scheme for surface crack.

The influence functions for stresses at γ due to dislocations at ω are given in the Appendix. In the limits $\gamma \to 0$ and $\omega \to 0$ the denominators of the influence functions vanish. Since the numerators of $\Gamma^p_{yy}(\gamma \to 0, \omega \to 0)$ and $\Gamma^p_{xy}(\gamma \to 0, \omega \to 0)$ also vanish, these functions are found to give finite contributions to the stress at $\gamma = 0$; since the numerators of $\Gamma^p_{xx}(\gamma \to 0, \omega \to 0)$ remain nonzero, these functions are found to give infinite contributions to the stress at $\gamma = 0$.

Assuming that the crack opening displacements are analytic, their derivatives, the dislocation densities, are also analytic and can be expressed as a power series in ω:

$$\mu_p(\omega) = \mu_{0p} + \sum_{\beta=1}^{\infty} \mu_{\beta p}\omega^{\beta}, \qquad p = 1, 2.$$

The expressions for stresses near the surface are then

$$\lim_{\gamma \to 0} \begin{Bmatrix} \sigma_{xx}(\gamma) \\ \sigma_{yy}(\gamma) \\ \sigma_{xy}(\gamma) \end{Bmatrix} = \lim_{\gamma \to 0} \sum_{p=1}^{2} \int_0^1 \mathrm{d}\omega \left[\mu_{0p} + \sum_{\beta=1}^{\infty} \mu_{\beta p}\omega^{\beta} \right] \begin{Bmatrix} \Gamma^p_{xx}(\gamma, \omega) \\ \Gamma^p_{yy}(\gamma, \omega) \\ \Gamma^p_{xy}(\gamma, \omega) \end{Bmatrix}. \tag{9}$$

The integrals of those terms containing ω^{β} in the expressions for σ_{yy} and σ_{xy} in eqn (9) are all zero for $\beta > 0$. The integrals

$$\lim_{\gamma \to 0} \int_0^1 \mathrm{d}\omega\, \mu_{0p} \begin{Bmatrix} \Gamma^p_{yy}(\gamma, \omega) \\ \Gamma^p_{xy}(\gamma, \omega) \end{Bmatrix} \equiv \mu_{0p} \begin{Bmatrix} I^p_{yy} \\ I^p_{xy} \end{Bmatrix}, \tag{10a}$$

(for $p = 1, 2$) are nonzero, namely

$$\begin{aligned} (I^1_{yy}, I^2_{yy}) &= \phi B_1(\tan \Psi, -1) \\ (I^1_{xy}, I^2_{xy}) &= \phi B_2(\tan \Psi, -1) \end{aligned} \tag{10b}$$

where $B_1 = -4\Psi \cos^3 \Psi$, $B_2 = 4\cos^2 \Psi(\Psi \sin \Psi - \cos \Psi)$ from which the stresses follow,

$$\lim_{\gamma \to 0} \begin{Bmatrix} \sigma_{yy}(\gamma) \\ \sigma_{xy}(\gamma) \end{Bmatrix} = \sum_{p=1}^{2} \mu_{0p} \begin{Bmatrix} I^p_{yy} \\ I^p_{xy} \end{Bmatrix}. \tag{11}$$

Since $y = 0$ is a free surface, $\sigma_{yy}(\gamma = 0) = \sigma_{xy}(\gamma = 0) = 0$. In order to have a continuous stress state at this point, the dislocation density must satisfy $\lim_{\gamma \to 0} \sigma_{yy}(\gamma) = \lim_{\gamma \to 0} \sigma_{xy}(\gamma) = 0$ also. Thus eqn (11) requires that

$$\mu_{01} \tan \Psi - \mu_{02} = 0. \tag{12}$$

The integrals of those terms containing ω^{β} in the expression for σ_{xx} in eqn (9) are all finite for $\beta > 0$. However, the integrals

$$\lim_{\gamma \to 0} \int_0^1 \mathrm{d}\omega \mu_{0p} \Gamma^p_{xx}(\gamma, \omega) \equiv \mu_{0p} I^p_{xx} \tag{13a}$$

(for $p = 1, 2$) give an infinite contribution to $\lim_{\gamma \to 0} \sigma_{xx}(\gamma)$:

$$\begin{aligned} I^1_{xx} &= -\phi\{B_3 \tan \Psi + B_4[-\cot 2\Psi + \tan \Psi \lim_{\gamma \to 0} \ln \gamma]\} \\ I^2_{xx} &= \phi\{B_3 + B_4[1 + \lim_{\gamma \to 0} \ln \gamma]\} \end{aligned} \tag{13b}$$

where $B_3 = 4\Psi \cos \Psi(2\cos^2 \Psi - \sin^2 \Psi)$, $B_4 = 8\cos^2 \Psi \sin \Psi$. If $\lim_{\gamma \to 0} \sigma_{xx}(\gamma)$ is to be finite, the condition which must be satisfied is again that of eqn (12). Thus, the vector of the dislocation density at the surface contributes only to opening of the crack and not to relative slippage of the faces.

Equation (12) is satisfied if both μ_{01} and μ_{02} are set equal to zero. Although this is more strict a condition than eqn (12) requires, it has been found to give the proper results computationally for the stress intensity factors; it seems to be only one of the many possible conditions which satisfy eqn (12) that could be chosen to provide the required closure conditions discussed after eqn (7). In terms of the nomenclature of eqn (7) the two additional equations for the surface crack are (for $\Psi \neq 0$)

$$f_p(t_n = -1) \simeq f_p(t_{nM_n}) = 0, \qquad p = 1, 2, \tag{14}$$

When $\Psi = 0$, the infinite integrals from eqn (9) vanish, as do all of the finite integrals except I^1_{xx} and I^2_{xy} which now both become equal to 4ϕ. Thus, for a surface crack normal to the free surface,

$$\lim_{\gamma\to 0}\begin{Bmatrix}\sigma_{xx}(\gamma)\\ \sigma_{yy}(\gamma)\\ \sigma_{xy}(\gamma)\end{Bmatrix} = 4\phi\begin{Bmatrix}\mu_{01}\\ 0\\ \mu_{02}\end{Bmatrix}. \tag{15}$$

The two additional equations for the surface crack perpendicular to the free surface are, therefore,

$$\mu_{01} = \sigma_{xx}(\gamma = 0)/4\phi \qquad \mu_{02} = \sigma_{xy}(\gamma = 0)/4\phi = 0. \tag{16a, b}$$

Equation (4) indicates that keeping $\mu_{01} = \mu_1(t = -1)$ finite requires $f_1(t = -1) = 0$. Similarly $\mu_{02} = \mu_2(t = -1) = 0$ gives $f_2(t = -1) = 0$. Expressing these conditions in terms of the nomenclature of eqn (7) again gives the requirements of eqn (14). For all surface cracks, then, the dislocation densities at the surface are set equal to zero.

Figure 3 compares the values for the stress intensity factors (computed as in eqn 18 below), obtained using this technique, with the results of Khrapkov[6], obtained through conformal mapping. The stress intensity factors are for cracks at various angles to the free surface of a semi-infinite region subjected to remotely applied shear and normal stresses. The number of collocation points used to obtain these results varied from $M_1 = 10$ to $M_1 = 150$, with larger angles and larger values of n requiring more points. The agreement with his results is excellent.

It should be mentioned here that an alternative method for reducing eqn (6) to eqn (7) was tried for surface cracks. The procedure of Refs. [4, 5], formally correct when the intervals of integration are $(-1, 1)$, was applied to the dimensionless variables γ and ω of Fig. 2, giving integration intervals of $(0, 1)$. The advantage in doing this is that the number of traction equations (two for each γ_r where r now ranges from 1 to $M/2$) is equal to the number of unknown values for the dislocation densities (two for each ω_k where $k = 1, 2, \ldots, M/2$). However, since the influence functions Γ^p_{yy} and Γ^p_{xy} are both zero for the surface point $\gamma_{M/2} = 0$, the equations for the normal and shear tractions at that point are not independent and an extra condition, again provided by eqn (12), is needed. The results for this method were found to be inferior to those of the technique just described (eqns 9–14).

Branch crack

For the case of a branch crack at the tip of an embedded crack, the two cracks must together contain the specified amount of entrapped dislocation and a second integral is added to eqn (8) to account for the dislocations of the branch crack.

The additional two equations to be used for the branch crack should at least ensure the readily provable result that the stress singularity at the point of crack intersection is less than that at a crack tip. This could be done by enforcing a continuous distribution of the dislocation density across the point of intersection, but it was found that somewhat more accurate results are obtained by setting the amplitude of the singular dislocation density of the branch crack equal to zero at the point where it intersects the main crack. This leads to the same condition as

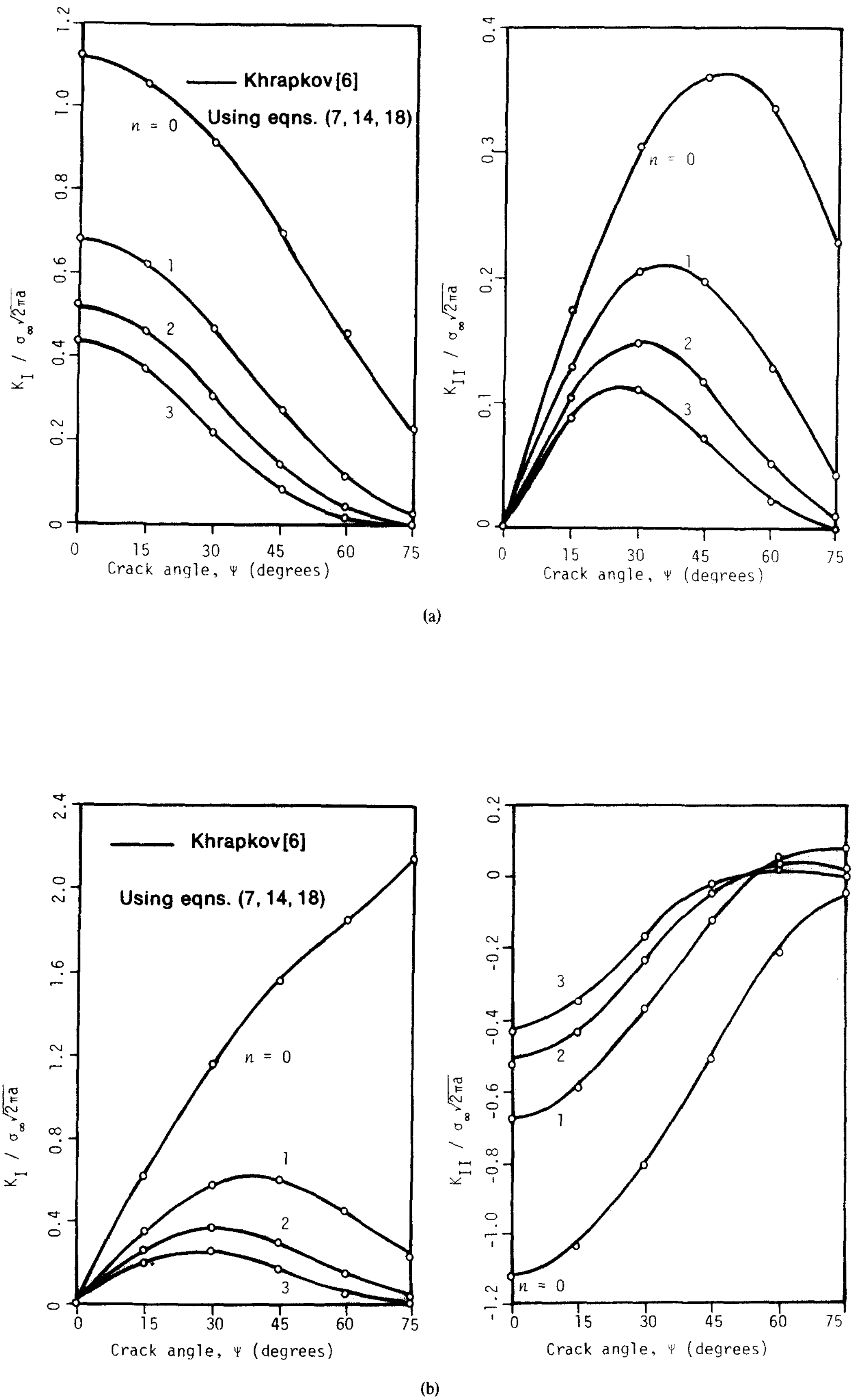

Fig. 3. Comparison of surface crack stress intensity factors obtained using eqns. (7), (14) and (18) with those of Khrapkov [6]. Crack geometry of Fig. 2, $\sigma_{xx} = \sigma_\infty y^n$.

eqn (14), this time applied to the branch crack. The resulting condition happens to coincide with that used by Lo[7] but he needs to model only the branch crack, since his influence function takes into account the boundary condition imposed by the presence of the main crack.

The results of the above procedure are compared with those of Lo in Fig. 4. Agreement is again very good. Between 20 and 120 collocation points on both the main crack and branch crack were used in obtaining these results, with larger a/a_b values requiring more points.

Equation (7), along with eqns (8) (or its modified version when a branch crack is present), and eqn (14) form a complete set of linear equations which can be solved for the $f_p(t_{nk})$. The stress σ_{jk} at any point (x, y) in the body can then be computed as

$$\sigma_{ij}(x, y) \simeq \sum_{n=1}^{N} \frac{\pi}{M_n} \sum_{p=1}^{2} \sum_{k=1}^{M_n} f_p(t_{nk}) \Gamma_{ij}^{p}([x, y], t_{nk}). \tag{17}$$

The stress intensity factors K_I and K_{II} are obtained directly from the values of f_p at the crack tip, K_I being proportional to the dislocation density component which tends to open the crack and K_{II} being proportional to the component tending to cause relative slippage between the sides of the crack. For instance, for a surface crack having an angle Ψ with the surface normal (see Fig. 1), the stress intensity factors take the form

$$\begin{aligned} K_I &= \pi^{3/2}\phi[-f_1(t_1)\cos\Psi - f_2(t_1)\sin\Psi]\sqrt{a} \\ K_{II} &= \pi^{3/2}\phi[-f_1(t_1)\sin\Psi + f_2(t_1)\cos\Psi].\sqrt{a}. \end{aligned} \tag{18}$$

Crack opening displacements are computed by integrating the dislocation densities μ_p and using Chebyshev integration formulae[11]:

$$\Delta u_p(t_s) - \Delta u_p(-1) = a\int_{-1}^{t_s} \mu_p(t)\,\mathrm{d}t \simeq \frac{a\delta_p}{M_n}\left(M_n - S + \frac{1}{2}\right) - \frac{2a}{M_n}\sum_{k=1}^{M_n} f_p(t_k) \sum_{l=1}^{M_n-1} \frac{1}{l}$$
$$\times \cos\left[\frac{l\pi(2k-1)}{2M_n}\right]\sin\left[\frac{l\pi(2S-1)}{2M_n}\right] \tag{19}$$

where Δu_p denotes the opening displacement in the p-direction.

INFINITE CRACK ARRAYS

In geometries containing an infinite number of identical cracks which have a common spacing of s, it is only necessary to solve for one generic crack, since all cracks have the same dislocation distribution. For example, with the array of cracks lying along the x-axis, the tractions for the crack $m = 0$ are, from eqn (7),

$$\begin{Bmatrix} \sigma_\nu(\zeta_{0r}) \\ \sigma_\tau(\zeta_{0r}) \end{Bmatrix} = \frac{\pi}{M} \sum_{n=-\infty}^{\infty} \sum_{p=1}^{2} \sum_{k=1}^{M} f_p(t_{nk}) \begin{Bmatrix} \Gamma_\nu^p(\zeta_{0r}, t_{nk}) \\ \Gamma_\tau^p(\zeta_{0r}, t_{nk}) \end{Bmatrix} \tag{20}$$

where crack n is located a distance ns along the x-axis from the crack $n = 0$.

For sufficiently large n, say $|n| > n^*$, the influence functions $\Gamma_{jk}^p(\zeta_{0r}, t_{nk})$, for a particular r and k, will depend only on ξ, the x-component of the distance between ζ_{0r} and t_{nk}. The asymptotic behavior of the influence functions takes the form $\xi^{-\epsilon}$, as shown in Table 1. The summation on n in eqn (20) is performed explicitly for $|n| \le n^*$. The remaining terms are evaluated by the use of the Euler–Maclaurin summation formula[8, No. 3.6.28].

The stress intensity factors for infinite arrays of equally-spaced cracks in infinite and semi-infinite media, subjected to mode I loading, are compared in Fig. 5 with results from Tada *et al.*[9]. No more than forty collocation points along the crack were used to obtain the results shown. Agreement is within the 2% accuracy quoted for the results of Ref. [9].

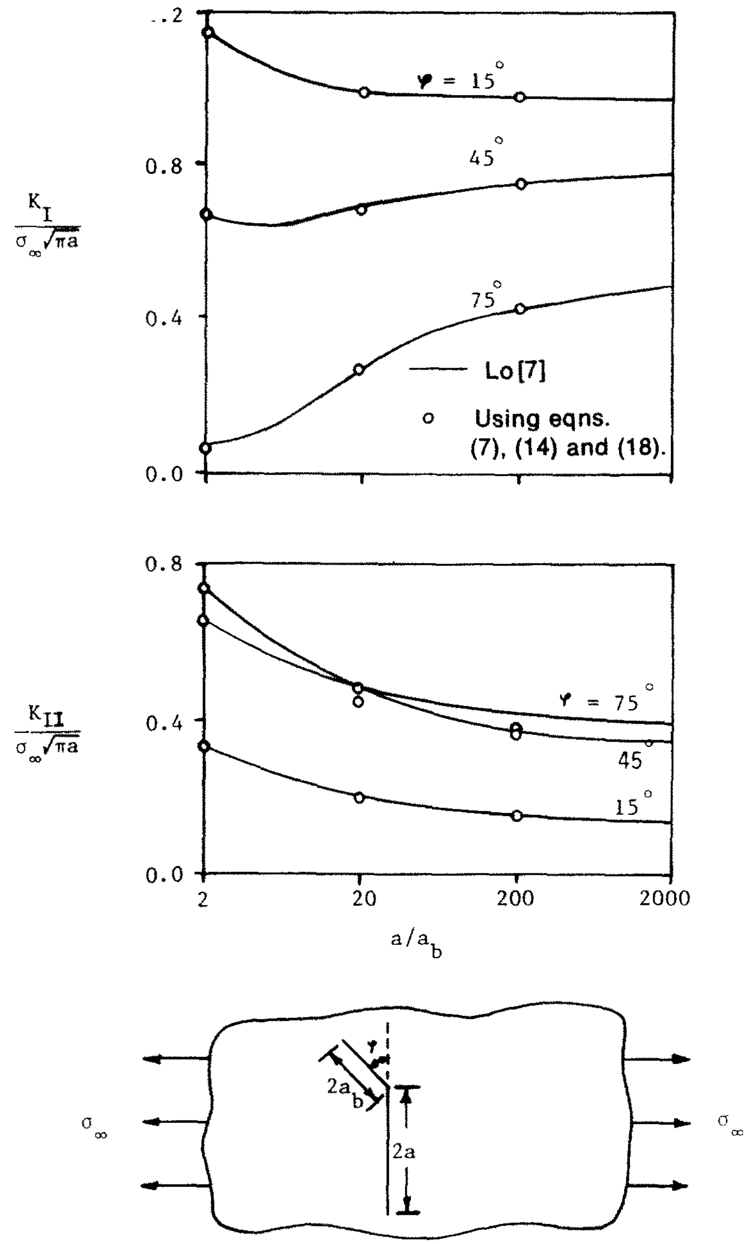

Fig. 4. Comparison of branch crack stress intensity factors with those of Lo [7].

Table 1. Asymptotic behavior of $\Gamma^{\xi}_{jk}([x, y], [x_d, y_d])$ for $|\xi| = |x_d - x| \gg |y \pm y_d|$. Table gives the values of ϵ for the asymptotic form $\Gamma \sim \xi^{-\epsilon}$

	Infinite body	Semi-infinite body (free surface at $y = 0$)
Γ^1_{xx}	2	2
Γ^2_{xx}	1	3
Γ^1_{yy}	2	4
Γ^2_{yy}	1	3
Γ^1_{xy}	1	3
Γ^2_{xy}	2	4

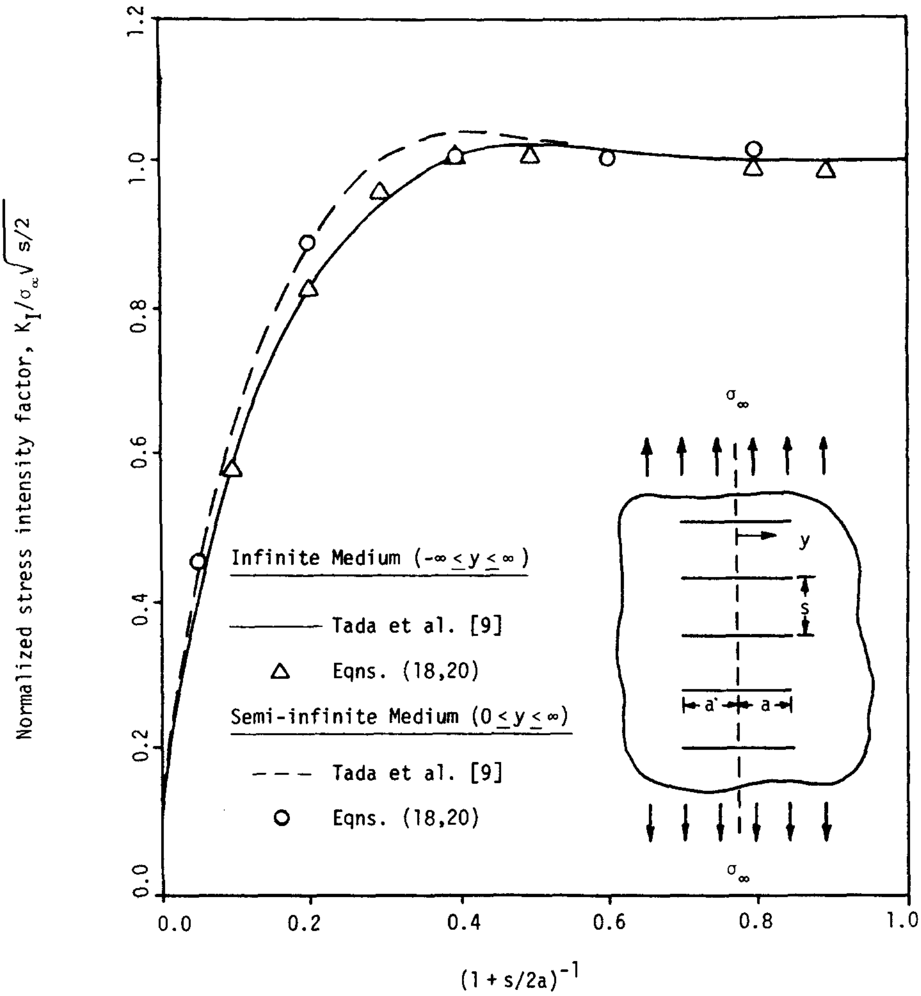

Fig. 5. Comparison of stress intensity factors obtained for infinite array of cracks in semi-infinite medium with those from Tada *et al.*[9].

CONCLUSIONS

The well-known equivalence of cracks and distributions of edge dislocations have been used to formulate integral expressions for crack tractions. For two-dimensional problems, approximate integration formulae employing Chebyshev polynomials were used to reduce these equations to a system of $2\sum_{m=1}^{N}(M_m - 1)$ linear algebraic equations for the crack tractions in terms of $2\sum_{n=1}^{N} M_n$ unknown dislocation densities. For unbranched cracks which are completely embedded, expressions for the entrapped dislocations serve to close the set of equations. For a crack intersecting a free surface, it was found that the dislocation at the point of intersection has only an opening component and no sliding component. Setting both of these components to zero at the surface was shown to give values for the stress intensity factors that are virtually identical to those obtained through conformal mapping[6]. For a branch crack, it was found that setting the dislocation density nearest the point of intersection equal to zero gives results which agree with those from other similar numerical techniques[7].

Extension of these procedures to infinite arrays of cracks is straightforward. The infinite number of cracks is explicitly considered by using asymptotic expresssions for the dislocation influence functions in conjunction with the Euler–Maclaurin summation formula. The results for two problems for which previously published results are available compare very well with those values[9].

A listing of the Fortran computer program used to implement the numerical techniques described in this paper is available in Ref. [10].

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APPENDIX

Edge dislocation influence functions

For any generic surface crack (e.g. as shown in Fig. 2) the stress σ_{jk} at γ due to a normalized unit edge dislocation ($b_p/a = 1$) at ω is given by $\Gamma^p_{jk}(\gamma, \omega)$. Defining $c_n = \cos(n\Psi)$, and $X = \gamma^2 + 2\gamma\omega\cos(\Psi) + \omega^2$, the influence functions for an isotropic homogeneous medium are:

$$\Gamma^1_{xx}(\gamma,\omega) = \frac{\phi c_1}{2}\left\{\frac{c_2-2}{\gamma-\omega} + \frac{1}{X^2}[(2-c_2)\gamma^3 + (3c_2)\gamma^2\omega + (3c_2)\gamma\omega^2 + (2-c_2)\omega^3] + \frac{1}{X^3}[(2c_4)\gamma^4\omega + (6c_2-2c_4)\gamma^3\omega^2 + (6-c_2)\gamma^2\omega^3 + (2c_2-6)\gamma\omega^4 - (2c_2)\omega^5]\right\}$$

$$\Gamma^2_{yy}(\gamma,\omega) = \frac{\phi\sin\Psi}{2}\left\{\frac{c_2}{\gamma-\omega} + \frac{1}{X^2}[(-c_2)\gamma^3 + (c_2-2)\gamma^2\omega + (2-c_2)\gamma\omega^2 + c_2\omega^3] + \frac{1}{X^3}[(-2c_4-2c_2)\gamma^4\omega + (4c_4+2c_2-2)\gamma^3\omega^2 + (6c_2+6)\gamma^2\omega^3 - (2c_4+2c_2)\gamma\omega^4 - (4c_2+4)\omega^5]\right\}$$

$$\Gamma^1_{xy}(\gamma,\omega) = \frac{\phi c_1}{2}\left\{\frac{-c_2}{\gamma-\omega} + \frac{1}{X^2}[(c_2)\gamma^3 + (2+c_2)\gamma^2\omega + (2+c_2)\gamma\omega^2 + (c_2)\omega^3] - \frac{1}{X^3}[(2c_4+4c_2)\gamma^4\omega + (2c_4+6c_2+12)\gamma^3\omega^2 + (6+18c_2)\gamma^2\omega^3 + (4c_4+2c_2+6)\gamma\omega^4 + (2c_2)\omega^5]\right\}$$

$$\Gamma^2_{xy}(\gamma,\omega) = \frac{\phi\sin\Psi}{2}\left\{\frac{-2-c_2}{\gamma-\omega} + \frac{1}{X^2}[(2+c_2)\gamma^3 + (3c_2)\gamma^2\omega - (3c_2)\gamma\omega^2 - (2+c_2)\omega^3] + \frac{1}{X^3}[(2c_4+6c_2+4)\gamma^4\omega + (6c_2+6)\gamma^3\omega^2 - (6c_2+6)\gamma^2\omega^3 - (2c_4+6c_2+4)\gamma\omega^4]\right\}$$

$$\Gamma^1_{xy}(\gamma,\omega) = \frac{\phi\sin\Psi}{2}\left\{\frac{c_2}{\gamma-\omega} - \frac{1}{X^2}[(c_2)\gamma^3 + (2-c_2)\gamma^2\omega + (c_2-2)\gamma\omega^2 - (c_2)\omega^3] + \frac{1}{X^3}[(2c_4+6c_2+4)\gamma^4\omega + (6c_2+6)\gamma^3\omega^2 - (6c_2+6)\gamma^2\omega^3 - (2c_4+6c_2+4)\gamma\omega^4]\right\}$$

$$\Gamma^2_{xy}(\gamma,\omega) = \frac{\phi c_1}{2}\left\{\frac{-c_2}{\gamma-\omega} + \frac{1}{X^2}[(c_2)\gamma^3 + (2+c_2)\gamma^2\omega + (2+c_2)\gamma\omega^2 + (c_2)\omega^3] + \frac{1}{X^3}[(2c_4)\gamma^4\omega + (6c_2-2c_4)\gamma^3\omega^2 + (6-6c_2)\gamma^2\omega^3 + (2c_2-6)\gamma\omega^4 - (2c_2)\omega^5]\right\}.$$